

# Quantum Decoherence of Phonons in Bose-Einstein Condensates

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We apply recently developed techniques from quantum optics and quantum information science to Bose-Einstein Condensates (BECs) in order to study the quantum decoherence of phonons of isolated BECs. In the last few years, major advances in the manipulation and control of phonons have highlighted their potential as carriers of quantum information in quantum technologies, particularly in quantum processing and quantum communication. Although most of these studies have focused on trapped ion and crystalline systems, another promising system that has remained relatively unexplored is BECs. The potential benefits in using this system have been emphasized recently with proposals of relativistic quantum devices that exploit quantum states of phonons of BECs to achieve, in principle, superior performance over standard non-relativistic devices. Quantum decoherence is often the limiting factor in the practical realization of quantum technologies but we show that this is only expected to heavily constrain the performance of these phononic BEC devices at very high phonon frequencies, being negligible at lower frequencies.

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## INTRODUCTION

Quantum decoherence, the environment-induced dynamical destruction of quantum coherence, has a wide range of applications. For example, it plays an important role in fundamental questions of quantum mechanics where it continues to provide insights into a potential resolution of the non-observation of quantum superpositions at macroscopic scales [1, 2]. Furthermore, its control is imperative to the operation and physical realization of quantum technologies that could soon revolutionize our technological world. This is because the time in which a quantum system decoheres is often found to be much shorter than the time that characterizes its energy relaxation [3–6].

BECs are considered to be promising candidates for the implementation of various quantum technologies since they can usually be well-isolated from their surroundings and so offer relatively long coherence times. In particular, two-component Bose-Einstein condensates, either as bosons condensed in two different spatial sites [7, 8] or as condensed bosons in different hyperfine levels [9, 10], have been widely considered for the implementation of certain quantum technologies. A primary application for such two-component BECs is quantum metrology [11–17], such as atomic clocks and accelerometers, but other applications, such as quantum computation and communication, have also been investigated [17–22]. In particular, a major advancement in this area has been the development of BECs on atom chips, which facilitates the control of many BECs [23–25].

Recently, BECs have been considered in quantum technology from a fundamentally different viewpoint. In

[26–28], instead of the atomic states of the BECs being used as the carriers of quantum information, the collective quantum excitations of the atoms, phonons, were considered instead. In the last few years, the emerging field of phonon-based quantum technologies has attracted considerable attention [29–35]. For example, in [30] a phononic equivalent of circuit-QED was considered for quantum processing, and in [31] phononic quantum networks were proposed. Phonons behave in similar ways to photons but their advantages include the fact that they can be localized and made to interact with each other while still maintaining long coherence times [35]. They also have, in general, much shorter wavelengths than the photons that are created in laboratory settings, allowing for regimes of atomic physics to be explored which cannot be reached in photonic systems.

Phonon-based quantum technologies have principally concentrated on crystalline and ion trap systems rather than BEC systems. However, the devices proposed in [26–28] illustrate the potential benefits of BEC systems. These proposed devices exploit the fact that phonons behave similarly to photons but propagate with much slower speeds. This has already been utilized in analogue gravity experiments and has culminated in the recent observation of an acoustic analogue of the elusive Hawking radiation of a black hole where, in this case, it is sound waves rather than light waves that cannot escape [36–38]. However, in [26, 27] it was demonstrated that the slow propagation speeds can also be used to enhance real rather than just analogue spacetime effects, allowing for the development of quantum metrology devices. For example, a gravitational wave (GW) detector was proposed in [27] where the GW modifies the phononic field in a

measurable way since the slow propagation speeds allow for a resonance process in a micrometre sized system for promising GW signals. Interestingly, initial calculations suggest that these phononic quantum devices should have a precision that is, in principle, orders of magnitude superior to the state-of-the-art [27, 28].

However, understanding the quantum decoherence of phonons in BECs will be crucial in assessing the practical realization of these proposed phononic quantum technologies that utilise BECs. Although quantum decoherence has been considered for the condensed atoms of a BEC, it has not, as far as we are aware, been considered for the phonons of BECs. Instead, previous studies have concentrated on the energy relaxation time of the phonons in isolated [39–61] and open BEC systems [62], which can be used to set only an upper bound for the time scale of quantum decoherence.

Here we consider the quantum decoherence processes for a single-mode phononic state of an isolated BEC. We start with a quantum field theory description of the BEC from which the phonons and their interactions can be derived. After determining how the reduced density matrix of a single-mode phononic state evolves in the Born-Markov approximation, we then employ techniques from quantum optics and quantum information science to determine how certain global entropic measures and nonclassical indicators evolve for this state when it is of Gaussian form, which can be used to quantify quantum decoherence. For illustrative purposes, we apply these general techniques to the specific case of a three-dimensional and uniform version of the phononic-based GW detector proposed in [27], and find that, although quantum decoherence will affect the performance of the device at very high phonon frequencies, the effects at lower frequencies are suppressed.

## I. TIME EVOLUTION OF THE PHONON DENSITY OPERATOR

### A. Phonons in the Bogoliubov Approximation

The quantum field Hamiltonian for a rarefied, interacting, non-relativistic Bose gas is (see e.g. [63]):

$$\begin{aligned} \hat{H} = & \int d\mathbf{r} \hat{\psi}^\dagger(\mathbf{r}) \left[ -\frac{\hbar^2}{2m} \nabla^2 + \mathcal{V}(\mathbf{r}) \right] \hat{\psi}(\mathbf{r}) \\ & + \frac{1}{2} \int d\mathbf{r} d\mathbf{r}' \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}') \mathcal{U}(\mathbf{r}' - \mathbf{r}) \hat{\psi}(\mathbf{r}) \hat{\psi}(\mathbf{r}'), \end{aligned} \quad (1)$$

where  $\hat{\psi}^\dagger(\mathbf{r})$  and  $\hat{\psi}(\mathbf{r})$  are the field operators creating and annihilating a bosonic atom at position  $\mathbf{r}$ ;  $\mathcal{U}(\mathbf{r})$  is the two-body potential; and  $\mathcal{V}(\mathbf{r})$  is the trap potential. For a uniform gas occupying a volume  $V$ , we can substitute the plane-wave solutions  $\hat{\psi}(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} \hat{a}_{\mathbf{p}} e^{i\mathbf{p} \cdot \mathbf{r}/\hbar}$  with  $\mathbf{p} = 2\pi\hbar\mathbf{n}/L$ ;  $\mathbf{n} = (n_x, n_y, n_z)$ ; and  $n_x, n_y, n_z \in \mathbb{N}$ ; to obtain

a complete description of the gas in terms of annihilation and creation operators in momentum space:

$$\hat{H} = \sum_{\mathbf{p}} \frac{p^2}{2m} \hat{a}_{\mathbf{p}}^\dagger \hat{a}_{\mathbf{p}} + \frac{1}{2V} \sum_{\mathbf{p}, \mathbf{p}', \mathbf{q}} \mathcal{U}_{\mathbf{q}} \hat{a}_{\mathbf{p}+\mathbf{q}}^\dagger \hat{a}_{\mathbf{p}'-\mathbf{q}}^\dagger \hat{a}_{\mathbf{p}'} \hat{a}_{\mathbf{p}}, \quad (2)$$

where  $\mathcal{U}_{\mathbf{q}} = \int \mathcal{U}(\mathbf{r}) e^{-i\mathbf{q} \cdot \mathbf{r}/\hbar} d\mathbf{r}$ .

Assuming the condensate to be macroscopically occupied, we next apply the Bogoliubov approximation [64]. This involves replacing  $\hat{a}_0$  and  $\hat{a}_0^\dagger$  with the c-number  $\sqrt{N_0}$  and only retaining terms quadratic in  $a_{\mathbf{p} \neq 0}$  and  $a_{\mathbf{p} \neq 0}^\dagger$  (the higher-order terms are suppressed since they have fewer factors of  $\sqrt{N_0} \gg 1$ ), where  $N_0$  is the number of atoms in the condensate. In this approximation one also replaces the microscopic potential  $\mathcal{U}$  with an effective soft potential and expands the  $\mathbf{q} = 0$  component up to quadratic terms in the coupling constant  $g = 4\pi\hbar^2 a/m$  where  $a$  is the s-wave scattering length and  $m$  is the atomic mass [63, 65].

The resulting Hamiltonian in  $a_{\mathbf{p} \neq 0}$  and  $a_{\mathbf{p} \neq 0}^\dagger$  can then be diagonalized by applying the following Bogoliubov transformation (see e.g. [63]):

$$\hat{a}_{\mathbf{p}} = u_p \hat{b}_{\mathbf{p}} + v_p \hat{b}_{-\mathbf{p}}^\dagger, \quad (3)$$

$$\hat{a}_{\mathbf{p}}^\dagger = u_p \hat{b}_{\mathbf{p}}^\dagger + v_p \hat{b}_{-\mathbf{p}}, \quad (4)$$

to obtain:

$$\hat{H} = \epsilon_0 + \sum_{\mathbf{p} \neq 0} \hbar\omega_p \hat{b}_{\mathbf{p}}^\dagger \hat{b}_{\mathbf{p}}, \quad (5)$$

where  $\hat{b}_{\mathbf{p}}^\dagger$  and  $\hat{b}_{\mathbf{p}}$  are the creation and annihilation operators for quasi-particles;  $\epsilon_0$  is their ground state energy;  $c_s$  and  $\omega_p$  are respectively the speed of sound and quasi-particle frequency:

$$\hbar\omega_p := \sqrt{c_s^2 p^2 + \left(\frac{p^2}{2m}\right)^2}, \quad (6)$$

$$c_s := \sqrt{\frac{gn}{m}}, \quad (7)$$

and  $u_p$  and  $v_p$  must satisfy:

$$u_p, v_p = \pm \left( \frac{p^2/2m + gn}{2\hbar\omega_p} \pm \frac{1}{2} \right)^{\frac{1}{2}}, \quad (8)$$

where  $n$  is the number density of the gas.

As illustrated by (5), the Bose gas in this Bogoliubov approximation can be described by a non-interacting gas of quasi-particles, where the low momentum modes ( $p \ll mc$ ) are phonons travelling at speed  $c_s$  since  $\hbar\omega_p \approx c_s p$  in this regime. Given that we have an ideal gas of quasi-particles, in thermal equilibrium the average occupation number  $N_{\mathbf{p}}$  of quasi-particles carrying momentum  $\mathbf{p}$  must satisfy:

$$N_{\mathbf{p}} := \langle \hat{b}_{\mathbf{p}}^\dagger \hat{b}_{\mathbf{p}} \rangle = \frac{1}{e^{\beta_p} - 1}, \quad (9)$$

where  $\beta_p := \hbar\omega_p/k_B T$ .

## B. Phonon Interactions

In the Bogoliubov approximation the phonons are non-interacting and so have infinite lifetimes. However, this approximation only keeps terms that are quadratic in  $\hat{a}_{\mathbf{p}}$  and  $\hat{a}_{\mathbf{p}}^\dagger$ . If we instead also include the (more suppressed) cubic and quartic terms then, after the above Bogoliubov transformation (3), these terms will provide interactions between the quasi-particles, resulting in finite lifetimes.

Concentrating on just a single momentum mode  $\mathbf{q}$  of the phonons, the cubic interaction terms for this mode are [46, 48, 55]:

$$\hat{H}_I = \hat{b}_{\mathbf{q}} \hat{E}_{\mathbf{q}}^\dagger + \hat{b}_{\mathbf{q}}^\dagger \hat{E}_{\mathbf{q}}, \quad (10)$$

where:

$$\hat{E}_{\mathbf{q}} := \hat{A} + \hat{B} + \hat{L}, \quad (11)$$

$$\hat{A}_{\mathbf{q}}^\dagger := g \sqrt{\frac{n}{V}} \sum_{\mathbf{p}, \mathbf{p}' \neq \{\mathbf{0}, \mathbf{q}\}} \mathcal{A}_{\mathbf{p}, \mathbf{p}'} \hat{b}_{\mathbf{p}} \hat{b}_{\mathbf{p}'}^\dagger \delta_{-\mathbf{q}, \mathbf{p} + \mathbf{p}'}, \quad (12)$$

$$\hat{B}_{\mathbf{q}}^\dagger := g \sqrt{\frac{n}{V}} \sum_{\mathbf{p}, \mathbf{p}' \neq \{\mathbf{0}, \mathbf{q}\}} \mathcal{B}_{\mathbf{p}, \mathbf{p}'} \hat{b}_{\mathbf{p}}^\dagger \hat{b}_{\mathbf{p}'}^\dagger \delta_{\mathbf{q}, \mathbf{p} + \mathbf{p}'}, \quad (13)$$

$$\hat{L}_{\mathbf{q}}^\dagger := g \sqrt{\frac{n}{V}} \sum_{\mathbf{p}, \mathbf{p}' \neq \{\mathbf{0}, \mathbf{q}\}} \mathcal{L}_{\mathbf{p}, \mathbf{p}'} \hat{b}_{\mathbf{p}} \hat{b}_{\mathbf{p}'}^\dagger \delta_{\mathbf{q}, \mathbf{p}' - \mathbf{p}}, \quad (14)$$

$$\mathcal{A}_{\mathbf{p}, \mathbf{p}'} := u_{\mathbf{q}}(v_{\mathbf{p}} v_{\mathbf{p}'} + u_{\mathbf{p}} v_{\mathbf{p}'} + v_{\mathbf{p}} u_{\mathbf{p}'}) + v_{\mathbf{q}}(u_{\mathbf{p}} v_{\mathbf{p}'} + v_{\mathbf{p}} u_{\mathbf{p}'} + u_{\mathbf{p}} u_{\mathbf{p}'}), \quad (15)$$

$$\mathcal{B}_{\mathbf{p}, \mathbf{p}'} := u_{\mathbf{q}}(u_{\mathbf{p}} u_{\mathbf{p}'} + v_{\mathbf{p}} u_{\mathbf{p}'} + u_{\mathbf{p}} v_{\mathbf{p}'}) + v_{\mathbf{q}}(v_{\mathbf{p}} v_{\mathbf{p}'} + v_{\mathbf{p}} u_{\mathbf{p}'} + u_{\mathbf{p}} v_{\mathbf{p}'}), \quad (16)$$

$$\frac{1}{2} \mathcal{L}_{\mathbf{p}, \mathbf{p}'} := u_{\mathbf{q}}(v_{\mathbf{p}} u_{\mathbf{p}'} + u_{\mathbf{p}} u_{\mathbf{p}'} + v_{\mathbf{p}} v_{\mathbf{p}'}) + v_{\mathbf{q}}(u_{\mathbf{p}} v_{\mathbf{p}'} + u_{\mathbf{p}} u_{\mathbf{p}'} + v_{\mathbf{p}} v_{\mathbf{p}'}). \quad (17)$$

The resonant interactions  $\hat{b}_{\mathbf{q}} \hat{L}_{\mathbf{q}}^\dagger$  and  $\hat{b}_{\mathbf{q}} \hat{B}_{\mathbf{q}}^\dagger$  are the Landau and Beliaev interactions [39–54]. In the Landau process  $\hat{b}_{\mathbf{q}} \hat{L}^\dagger$ , a quasi-particle from mode  $\mathbf{q}$  collides with a quasi-particle from another mode to create a higher-energy quasi-particle. Since this requires the thermal occupation of a quasi-particle mode, the process vanishes at zero temperature. On the other hand, in the Beliaev process  $\hat{b}_{\mathbf{q}} \hat{B}^\dagger$ , a quasi-particle of mode  $\mathbf{q}$  spontaneously annihilates into two new quasi-particles with lower energies, which is analogous to parametric down-conversion in quantum optics [66] and can occur at absolute zero.<sup>1</sup> Since the processes originate from (2), they can also be considered from the point-of-view of four-body interactions between the condensate and non-condensate atoms.

<sup>1</sup> However, due to the discretization of energy levels in trapping potentials, the Beliaev process is suppressed for the lowest energy modes.

## C. Markov Quantum Master Equation for a Single-Mode Phonon State

Treating the above single phonon mode  $\mathbf{q}$  as an open quantum system, and the rest of the quasi-particle modes as its environment, we can decompose the Hamiltonian for the full system  $\hat{H}$  in the following way:

$$\hat{H} = \hat{H}_S + \hat{H}_E + \hat{H}_I, \quad (18)$$

where  $\hat{H}_S$  and  $\hat{H}_E$  derive from (5), and respectively describe the free Hamiltonian of the considered single-mode phonon system and all other quasi-particle modes:

$$\hat{H}_S := \hbar \omega_{\mathbf{q}} \hat{b}_{\mathbf{q}}^\dagger \hat{b}_{\mathbf{q}}, \quad (19)$$

$$\hat{H}_E := \sum_{\mathbf{p} \neq \{\mathbf{0}, \mathbf{q}\}} \hbar \omega_{\mathbf{p}} \hat{b}_{\mathbf{p}}^\dagger \hat{b}_{\mathbf{p}}, \quad (20)$$

and  $\hat{H}_I$  is defined in (10). We explicitly ignore the interaction terms between the states of the large system  $E$ , which describes all quasi-particle modes with momentum  $\mathbf{p} \neq \mathbf{q} \neq 0$  and is assumed to be in thermal equilibrium.

To determine the evolution of the single-mode phonon system, we assume that the initial state of the full system is a product state  $\hat{\rho}(0) = \hat{\rho}_S(0) \otimes \hat{\rho}_E(0)$  and perform the Born-Markov approximation. That is, we assume that the coupling between  $E$  and  $S$  is weak, which is well justified for a rarefied Bose gas at low temperatures, and that the future evolution of  $\hat{\rho}_S(t)$  does not depend on its past history. The latter is satisfied when the environment correlation time  $\tau_E$  is much shorter than the time scale for significant change in  $S$ , which occurs when the environment is a large system maintained in thermal equilibrium, as expected here. The evolution of the single-mode system is then defined by the Lindblad master equation which, in diagonal form, is given by:

$$\frac{d\hat{\rho}_S}{dt} = -\frac{i}{\hbar} [\hat{H}'_S, \hat{\rho}_S] + \sum_{i=1}^4 \left( \hat{c}_i \hat{\rho}_S \hat{c}_i^\dagger - \frac{1}{2} \{ \hat{c}_i^\dagger \hat{c}_i, \hat{\rho}_S \} \right), \quad (21)$$

where  $\hat{H}'_S$  is the renormalized free Hamiltonian of the single-mode phonon system. Assuming the environment to be in thermal equilibrium, we have  $\hat{c}_1 = \sqrt{\gamma_1} \hat{b}_{\mathbf{q}}$ ,  $\hat{c}_2 = \sqrt{\gamma_2} \hat{b}_{\mathbf{q}}^\dagger$  where the rates  $\gamma_1$  and  $\gamma_2$  are related to environment correlation functions:

$$\gamma_1 = \frac{1}{\hbar^2} \int_{-\infty}^{\infty} dt' e^{i\omega_{\mathbf{q}} t'} \langle \tilde{E}_{\mathbf{q}}(t) \tilde{E}_{\mathbf{q}}^\dagger(t-t') \rangle_E, \quad (22)$$

$$\gamma_2 = \frac{1}{\hbar^2} \int_{-\infty}^{\infty} dt' e^{-i\omega_{\mathbf{q}} t'} \langle \tilde{E}_{\mathbf{q}}^\dagger(t) \tilde{E}_{\mathbf{q}}(t-t') \rangle_E, \quad (23)$$

and  $\tilde{E}_{\mathbf{q}}$  is the operator defined in (11) but now in the interaction picture. The master equation (21) therefore describes the time evolution of a single-mode quasi-particle of a BEC in thermal equilibrium when taking into account Landau and Beliaev interactions. It has also been derived in [55].

## II. TIME EVOLUTION OF THE PHONON COVARIANCE MATRIX

### A. The Covariance Matrix Formalism

By restricting our analysis to Gaussian states we can use the covariance matrix formalism to conveniently describe the dynamics of the single-mode phonon system. Such states are assumed in the relativistic quantum devices proposed in [26–28] that exploit the phonons of BECs, and can be readily created in BECs. For example, such states can be generated using Bragg spectroscopy [67–71] and in processes that are acoustic analogues to the dynamical Casimir effect [72, 73].

The covariance matrix formalism is often used in quantum optics and continuous-variable quantum information science, and is a convenient mathematical framework in quantum phase space for describing Gaussian states and their dynamics. It utilizes the fact that Gaussian states are completely defined by just their first  $\mathbf{d}$  and second  $\boldsymbol{\sigma}$  statistical moments:

$$d_i := \langle \hat{x}_i \rangle = \text{Tr}(\hat{x}_i \hat{\rho}_S), \quad (24)$$

$$\begin{aligned} \sigma_{ij} &:= \frac{1}{2} \langle \{\hat{x}_i, \hat{x}_j\} \rangle - \langle \hat{x}_i \rangle \langle \hat{x}_j \rangle \\ &\equiv \frac{1}{2} \text{Tr}(\{\hat{x}_i - d_i, \hat{x}_j - d_j\} \hat{\rho}_S), \end{aligned} \quad (25)$$

where  $\hat{x}_i$  are the quadrature phase space operators which, for the single-mode system, are defined as:

$$\hat{\mathbf{x}} := \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} := \frac{1}{2\kappa} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} \hat{b}_q \\ \hat{b}_q^\dagger \end{pmatrix}.$$

These phase space operators then obey the commutation relations  $[\hat{x}_i, \hat{x}_j] = \frac{i}{2\kappa^2} \Omega_{ij}$  where  $\Omega$  is of symplectic form, and  $\kappa$  is a constant that is often taken to be  $1/\sqrt{2}$  in quantum optics. By assuming Gaussian states we have, therefore, reduced the complete description of the state down from the infinite dimensional space of the density matrix to terms of the two-dimensional matrices  $\boldsymbol{\sigma}$  and  $\mathbf{d}$ , which are referred to as the covariance matrix and displacement matrix of the state respectively.

### B. Time Evolution of Statistical Moments

To determine how the covariance and displacement matrices evolve for the single-mode phonon system, we first transform the Lindblad equation for the density matrix (21) into the phase space basis  $\hat{x}_i$  and then use this equation in differential versions of (24) and (25) [74–76]. In fact it is possible to derive the time evolution equations for the covariance and displacement matrices of a general  $M$ -dimensional Gaussian state whose density matrix obeys (21): taking a general quadratic Hamiltonian<sup>2</sup>

$\hat{H}_S = H_0 + \kappa \hat{\mathbf{x}}^T \mathbf{H}_1 + \kappa^2 \hat{\mathbf{x}}^T \mathbf{H}_2 \hat{\mathbf{x}}$  where  $H_0$  is a constant;  $\mathbf{H}_1$  is a  $2M$ -dimensional column-vector; and  $\mathbf{H}_2$  is a  $2M \times 2M$  real-symmetric matrix, it is straightforward to show that the covariance and displacement matrices evolve in the following way:

$$\frac{d\mathbf{d}}{dt} = \boldsymbol{\mathcal{H}}_1 + \mathbf{A}\mathbf{d}, \quad (26)$$

$$\frac{d\boldsymbol{\sigma}}{dt} = \mathbf{A}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{A}^T + \mathbf{D}, \quad (27)$$

where  $\boldsymbol{\mathcal{H}}_1 := \Omega \mathbf{H}_1 / \hbar$ . These equations have been derived previously in quantum optics for the case when  $\mathbf{H}_1 = 0$  [74–76, 78]. In these quantum optics studies, the matrix  $\mathbf{A}$  and symmetric matrix  $\mathbf{D}$  are referred to as the drift and diffusion matrices respectively [76]. They are defined as:

$$\mathbf{D} := \frac{1}{4\kappa^4} \Omega \text{Re}(C^\dagger C) \Omega^T, \quad (28)$$

$$\mathbf{A} := \boldsymbol{\mathcal{H}}_2 + \mathbf{K}, \quad (29)$$

where:

$$\mathbf{K} := \frac{1}{2\kappa^2} \Omega \text{Im}(C^\dagger C), \quad (30)$$

$$\boldsymbol{\mathcal{H}}_2 := \frac{1}{\hbar} \Omega \mathbf{H}_2, \quad (31)$$

and the matrix  $\mathbf{C}$  is defined by  $\hat{c}_i = C_{ij} \hat{x}_j$ .

The general solution of (26) is:

$$\mathbf{d}(t) = \mathbf{X}(t) \mathbf{d}_0 + \mathbf{Y}(t), \quad (32)$$

where when  $\mathbf{A}$  is independent of time:

$$\mathbf{X}(t) = e^{\mathbf{A}t}, \quad (33)$$

$$\begin{aligned} \mathbf{Y}(t) &= \int_0^t e^{\mathbf{A}(t-s)} d\mathbf{s} \mathbf{H}_1 \\ &= \mathbf{A}^{-1} (e^{\mathbf{A}t} - 1) \mathbf{H}_1, \end{aligned} \quad (34)$$

so that the evolution of the displacement matrix is given by:

$$\mathbf{d}(t) = e^{\mathbf{A}t} \mathbf{d}_0 + \int_0^t e^{\mathbf{A}(t-s)} d\mathbf{s} \mathbf{H}. \quad (35)$$

Note that  $\mathbf{X}(t)$  has to fulfil the condition  $\lim_{t \rightarrow \infty} \mathbf{X}(t) = 0$  and so  $\mathbf{A}$  must have only eigenvalues with negative parts [74, 79].

The equation of motion for the covariance matrix (27) is, in general, a time varying differential Lyapunov matrix equation, of which the general solution is [79]:

$$\boldsymbol{\sigma}(t) = \mathbf{X}(t) \boldsymbol{\sigma}_0 \mathbf{X}^T(t) + \mathbf{Z}(t), \quad (36)$$

where:

$$\mathbf{X}(t) = \boldsymbol{\Phi}(t, 0), \quad (37)$$

$$\mathbf{Z}(t) = \int_0^t \boldsymbol{\Phi}(t, s) \mathbf{D} \boldsymbol{\Phi}^T(t, s) d\mathbf{s}. \quad (38)$$

<sup>2</sup>  $\hat{H}_S$  must be at most quadratic to preserve Gaussianity [77].

When  $\mathbf{A}$  is independent of time,<sup>3</sup>  $\Phi(t, s) = e^{\mathbf{A}(t-s)}$  and so the solution to (27) can be written as:<sup>4</sup>

$$\sigma(t) = e^{\mathbf{A}t} \sigma_0 e^{\mathbf{A}^T t} + \int_0^t e^{\mathbf{A}(t-s)} \mathbf{D} e^{\mathbf{A}^T(t-s)} ds, \quad (39)$$

for which an analytical expression can be obtained when  $\mathbf{A}$  is diagonalizable [81]. As above,  $\mathbf{X}(t)$  has to fulfil the condition  $\lim_{t \rightarrow \infty} \mathbf{X}(t) = 0$  for a stable solution and so again  $\mathbf{A}$  must have only eigenvalues with negative parts [74, 79].

A connection can be made with the evolution of the displacement vector and covariance matrix generated by a Gaussian unitary by neglecting all dissipative effects ( $\mathbf{D} = 0$ ,  $\mathbf{A} = \mathbf{H}_2$ ). In this case, assuming for convenience that  $\mathbf{H}_2$  and  $\mathbf{H}_1$  are independent of time, the solutions (32) and (36) reduce to:

$$\mathbf{d}(t) = \mathbf{S} \mathbf{d}_0 + \mathbf{e}, \quad (40)$$

$$\sigma(t) = \mathbf{S} \sigma_0 \mathbf{S}^T, \quad (41)$$

where  $\mathbf{S} := e^{\mathbf{H}_2 t}$  is the symplectic transformation corresponding to the free unitary evolution of the system, and  $\mathbf{e} := \mathbf{B} \mathbf{H}_1$  is a real vector with  $\mathbf{B} := \mathbf{H}_2^{-1}(e^{\mathbf{H}_2 t} - 1)$ . The matrix  $\mathbf{H}_2 = \Omega \mathbf{H}_2 / \hbar$  thus forms a symplectic algebra so that  $\mathbf{H}_2$  is a (real) Hamiltonian matrix and  $\mathbf{H}_2$  is a (real) symmetric matrix, which is also required by the Hermitian property of  $\hat{H}$ .

For our single-mode phonon system,  $\hat{H}_S$  is given by (19) and thus  $H_0 = 0$ ,  $\mathbf{H}_1 = \mathbf{0}$  and  $\mathbf{H}_2 = \hbar \omega'_q \mathbf{I}_2$  where  $\mathbf{I}_2$  is the two-dimensional identity matrix. The environment is also assumed to be in thermal equilibrium, and so the diffusion and drift matrices are given by:

$$\mathbf{D} = \frac{1}{4\kappa^2} \gamma_T \mathbf{I}_2, \quad (42)$$

$$\mathbf{A} = -\frac{1}{2} \gamma \mathbf{I}_2 + \omega'_q \Omega, \quad (43)$$

where  $\gamma_T := \gamma_1 + \gamma_2$ ;  $\gamma := \gamma_2 - \gamma_1$ ; and  $\omega'_q$  is the renormalized frequency of the single-mode phonon system.

Since the environment is in thermal equilibrium, the rates  $\gamma_1$  and  $\gamma_2$  are not independent but instead satisfy  $\gamma_1 = e^{\beta_q} \gamma_2$  [55, 82]. We can, therefore, write  $\gamma_T$  as:

$$\gamma_T := \gamma \coth\left(\frac{1}{2} \beta_q\right) = \gamma(1 + 2N_q^{th}), \quad (44)$$

where  $N_q^{th}$  is the average thermal occupation defined in (9). The matrix  $\mathbf{D}$  is then given by:

$$\mathbf{D} = \frac{(1 + 2N_q)}{4\kappa^2} \gamma \mathbf{I}_2 := \gamma \sigma_\infty, \quad (45)$$

where  $\sigma_\infty$  is the covariance matrix of a single-mode thermal state.

Substituting the above drift and diffusion matrices for the single-mode phonon system into the general time-independent solutions (32) (using (33) and (34)) and (39), the displacement vector and covariance matrix at time  $t$  are given by:

$$\mathbf{d}(t) = e^{-\frac{1}{2} \gamma t} \mathbf{R}(t) \mathbf{d}_0, \quad (46)$$

$$\sigma(t) = e^{-\gamma t} \left( \mathbf{R}(t) \sigma_0 \mathbf{R}^T(t) \right) + (1 - e^{-\gamma t}) \sigma_\infty, \quad (47)$$

where  $\mathbf{R}(t) := e^{\frac{1}{\hbar} \Omega \mathbf{H}_2 t}$  is the symplectic transformation corresponding to the free unitary evolution of the single-mode phonon system  $U = e^{-\frac{i}{\hbar} H_S t}$ , and thus  $\Omega \mathbf{H}_2$  forms the symplectic algebra (it is a Hamiltonian matrix).

Since  $\mathbf{H}_2 = \hbar \omega'_q \mathbf{I}_2$  for our phonon system,  $\mathbf{R}(t) = \cos(\omega_q t) \mathbf{I}_2 + \sin(\omega_q t) \Omega$ , which is just the usual symplectic (and in this case rotational) transformation for the phase shift operator. From (47), due to the dissipative effects, this free evolution is now damped by  $e^{-\gamma t}$  and the state asymptotically approaches  $\sigma_\infty$ :

$$\sigma(t) = e^{-\gamma t} \sigma_0 + (1 - e^{-\gamma t}) \sigma_\infty. \quad (48)$$

This form of equation has been derived in quantum optics studies [83–85] but the rate  $\gamma$  in those studies is not the same as that for the phononic system considered here. This rate comes from the expressions for  $\gamma_1$  and  $\gamma_2$  (22)–(23) given in Section IC. From these expressions, and assuming a continuum of modes, it can be easily shown that  $\gamma = \gamma_B + \gamma_L$  where:

$$\gamma_L := \frac{g^2 n}{V \hbar^2} \int_0^\infty \pi d\omega_k p_k \mathcal{L}_{kl}^2 \delta(\omega_q + \omega_k - \omega_l) (N_l^{th} - N_k^{th}), \quad (49)$$

$$\gamma_B := \frac{2g^2 n}{V \hbar^2} \int_0^\infty \pi d\omega_k p_k \mathcal{B}_{kl}^2 \delta(\omega_q - \omega_k - \omega_l) (1 + N_l^{th} + N_k^{th}), \quad (50)$$

which are just the Landau and Beliaev damping rates [39–55], where  $p_k$  is an assumed density of states. These rates have been explicitly calculated under various approximations. For example, in [48] the rates were calculated for a uniform three-dimensional BEC where it was found that, when  $k_B T \ll \hbar \omega_q$  such that Beliaev damping dominates over Landau damping  $\gamma_B \gg \gamma_L$ , the total damping rate  $\gamma$  can be approximated as [48]:

$$\gamma \approx \gamma_B \approx \frac{3}{640\pi} \frac{\hbar \omega_q^5}{m n c_s^5} \left( 1 + \left( \frac{k_B T}{\hbar \omega_q} \right)^3 \right), \quad (51)$$

which doesn't vanish at  $T = 0$ . On the other hand, in the opposite regime  $k_B T \gg \hbar \omega_q$ , Landau damping dominates over Beliaev damping, and for very high temperatures  $k_B T \gg \mu \gg \hbar \omega_q$  (where  $\mu = g n = m c_s^2$  is the chemical potential), the total damping rate was found to be given by [48]:

$$\gamma \approx \gamma_L \approx \frac{3\pi}{8} \frac{k_B T a}{\hbar c_s} \omega_q, \quad (52)$$

<sup>3</sup> There is no general analytic expression for the transition matrix  $\Phi(t, s)$  when  $\mathbf{A}$  is time dependent and in this case a numerical method is then the only way to obtain a solution [79].

<sup>4</sup> Another option is to convert (27) to a vector-valued ODE, which can then be readily solved, using vectorization of the matrix  $\sigma(t)$  [80].



whereas, for temperatures such that  $\mu \gg k_B T \gg \hbar\omega_q$ , the damping rate is found to be [48]:

$$\gamma \approx \gamma_L \approx \frac{3\pi^3}{8} \frac{(k_B T)^4}{mn\hbar^3 c_s^5} \omega_q. \quad (53)$$

Evolution equations for the displacement and covariance matrices in the form of (46)-(47) will apply to any single-mode Gaussian system with free Hamiltonian  $\hat{H}_S = \hbar\omega_q \hat{b}_q^\dagger \hat{b}_q$  that is interacting in the Born-Markov approximation with an environment that is in thermal equilibrium. Our specific case of a phonon in a BEC only manifests itself through the damping rates (49)-(50), which are the Landau and Beliaev damping rates for phonons, as well as the value of  $\omega_q$  for the phonons and the temperature  $T$  for the BEC.

It is important to note that, even though the interaction Hamiltonian for our phonon system (10) is cubic in field modes, Gaussianity of the state will be preserved under the Born-Markov approximation. This is evident from the fact that the general solution to (27) has the form of the transformation brought about by a general Gaussian channel, which is a trace-preserving completely positive map that maps Gaussian trace-class operators onto Gaussian trace-class operators [86–96]. It is also illustrated by studying the Fokker-Planck equation for the Wigner function that derives from a general Markov master equation for the density operator, where it is found that the Wigner function continues to be Gaussian [83, 84]. The evolution of a Gaussian state in the Markov approximation is generally analysed for an interaction Hamiltonian that is bilinear in the field operators (e.g. see [83, 84, 97]). However, the above analysis for a single-mode phonon state of a BEC clearly emphasizes that this isn't a necessary condition for the preservation of Gaussianity. This is essentially because, in the assumed Born-Markov approximation, the master equation for the phonon system (21) corresponds to a model in which the nonlinear terms are effectively not present (there is an effective linearization of the interaction Hamiltonian over the environmental field operators). This is important since it means that, under realistic approximations, the simple description of the displacement and covariance matrices fully characterizing the state will persist throughout the state's evolution.

### III. QUANTIFYING THE QUANTUM DECOHERENCE AND RELAXATION OF PHONONS OF BECS

The quantum decoherence of quantum optics systems has been extensively studied. In particular, for single-mode Gaussian states, the evolution of such states of electromagnetic radiation in thermal reservoirs was investigated in [98, 99] where certain properties, such as purity and squeezing of a displaced squeezed thermal mode, were analysed. Furthermore, in [85, 100], the quantum decoherence of a Gaussian quantum optics system was

characterized by analysing the evolution of certain global entropic measures and nonclassical indicators using the covariance matrix formalism. We can apply the same quantum optical techniques to our phononic system since the phonon system is in a Gaussian state and the evolution of the covariance matrix (48) takes the same form as in [85, 100], with only the damping rates being different. This similarity in the evolution of the state, despite the interactions being very different, is due to applying the Born-Markov approximation in both cases such that the quantum master equation for the phonons (21) is of a similar form to that of a quantum optics master equation as described in the previous section.

#### A. Evolution of Purity

For a single-mode Gaussian state, the purity  $\mu := \text{Tr}(\rho_S^2)$  is given by:

$$\begin{aligned} \mu &= \frac{1}{4\kappa^2 \sqrt{\det \sigma}} \\ &= \frac{1}{4\kappa^2 s} \\ &= \frac{1}{1 + 2\mathcal{N}}, \end{aligned} \quad (54)$$

where  $\mathcal{N}$  is the so-called “thermal” occupation of the state and  $s$  is its symplectic eigenvalue (the eigenvalue of the matrix  $|i\Omega\sigma|$ ). Using Williamson's theorem [101], any single-mode covariance matrix can be written in the general form [83, 84, 102]:

$$\sigma = \frac{1}{4\kappa^2 \mu} \begin{pmatrix} \cosh 2r + \sinh 2r \cos \psi & \sinh 2r \sin \psi \\ \sinh 2r \sin \psi & \cosh 2r - \sinh 2r \cos \psi \end{pmatrix}, \quad (55)$$

where  $r$  and  $\psi$  are defined by  $\xi = r e^{i\psi}$ , which is the squeezing parameter for a single-mode squeezing transformation. Using this general form of the covariance matrix in the solution (47) of its equation of motion, the purity of the single-mode state is found to evolve as [85, 99]:

$$\begin{aligned} \mu(t) &= \mu_0 \left( e^{-2\gamma t} + \frac{\mu_0^2}{\mu_\infty^2} (1 - e^{-\gamma t})^2 \right. \\ &\quad \left. + \frac{2\mu_0}{\mu_\infty} e^{-\gamma t} (1 - e^{-\gamma t}) \cosh 2r_0 \right)^{-\frac{1}{2}}, \end{aligned} \quad (56)$$

where  $\mu_\infty = (1 + 2N_q^{th})^{-1} = \tanh(\frac{1}{2}\beta_q)$  is the purity of the thermal state  $\sigma_\infty$  that the covariance matrix asymptotically approaches. The purity will undergo a local minimum for squeezed states for which  $r_0 > \max[\mu_0/\mu_\infty, \mu_\infty/\mu_0]$  at [85, 99, 100]:

$$t_{min} = \frac{1}{\gamma} \ln \left( \frac{\frac{\mu_0}{\mu_\infty} + \frac{\mu_\infty}{\mu_0} - 2 \cosh(2r_0)}{\frac{\mu_0}{\mu_\infty} - \cosh(2r_0)} \right), \quad (57)$$

which can provide a good characterization of the decoherence time of such states since this represents the point

at which the state becomes most mixed, with any subsequent increase in the purity just reflecting the state being driven towards the state of the environment [85, 100]. To determine this decoherence time one just needs to know the initial purity and squeezing of the state; the temperature of the BEC (which defines  $\mu_\infty$ ); and the damping rate  $\gamma$ , which can be estimated using (51)-(53) for a uniform BEC. As an example, the quantum decoherence time for phonons in a particular BEC is calculated and presented in Section IV. This BEC is that assumed in [27, 28] and the results can therefore be used to predict a decoherence time for the phononic GW detector discussed in the Introduction.

### B. Evolution of Nonclassical Depth

An alternative characterization of the decoherence time can be provided by the nonclassical depth [103], which is a popular measure for quantifying the nonclassicality of a quantum state. This measure has the physical meaning of the number of thermal photons necessary to destroy the nonclassical nature of the quantum state [104]. For a general Gaussian state, the nonclassical depth  $\tau$  detects the state as nonclassical if a canonical quadrature exists whose variance is below  $1/2$  [85] and, for a single-mode Gaussian state, it is given by:

$$\tau = \max\left[\frac{1}{2}\left(1 - \frac{e^{-2r}}{\mu}\right), 0\right]. \quad (58)$$

From the time evolution of the covariance matrix (47), the nonclassical depth  $\tau$  of the single-mode can be shown to evolve as [85]:

$$\tau(t) = \frac{1}{2\mu_\infty} \left( e^{-\gamma t} \left( 1 - \frac{\mu_\infty}{\mu_0} e^{-2r_0} \right) + \mu_\infty - 1 \right). \quad (59)$$

Note that the time at which the state becomes “classical” ( $\tau = 0$ ) is given by:

$$t_{\tau=0} = \frac{1}{\gamma} \ln \left( \frac{1 - \frac{\mu_\infty}{\mu_0} e^{-2r_0}}{1 - \mu_\infty} \right), \quad (60)$$

providing an alternative description for quantifying the decoherence time of quantum states.

### C. Evolution of Squeezing

As illustrated by the general covariance matrix of a single-mode Gaussian state (49), such states are fully defined by their first-statistical moment, purity, and squeezing  $\xi = r e^{i\psi}$ . For a single-mode state whose covariance matrix evolves as in (47),  $\psi$  is a constant of motion and  $r$  evolves as [85]:

$$\cosh(2r(t)) = \mu(t) \left( e^{-\gamma t} \frac{\cosh 2r_0}{\mu_0} + \frac{1 - e^{-\gamma t}}{\mu_\infty} \right). \quad (61)$$

### D. Evolution of Average Occupation

The average occupation of a single mode state  $\mathbf{q}$  is defined as  $N_{\mathbf{q}} := \langle \hat{b}_{\mathbf{q}}^\dagger \hat{b}_{\mathbf{q}} \rangle$ . From the Lindblad master equation for the density operator of the single-mode state interacting with an environment in thermal equilibrium (21), it is easy to show that the average occupation of the single-mode phonon state evolves as:

$$N_{\mathbf{q}}(t) = e^{-\gamma t} N_{\mathbf{q}}(0) + (1 - e^{-\gamma t}) N_{\mathbf{q}}^{th}, \quad (62)$$

where  $N_{\mathbf{q}}^{th} := (e^{\beta\omega_{\mathbf{q}}} - 1)^{-1}$  is the thermal occupation of the mode at temperature  $T$ . Since the average occupation is simply related to the average energy of the state, one can characterize the relaxation of the state from its evolution. Furthermore, since the average occupation is also defined by  $N_{\mathbf{q}} = \kappa^2 \text{Tr}(\boldsymbol{\sigma}) + \kappa^2 \mathbf{d}^T \mathbf{d} - 1/2$ , a single-mode Gaussian state at a time  $t$  is fully defined by its average occupation  $N_{\mathbf{q}}$ , purity  $\mu$  and squeezing  $r$ .

## IV. EXAMPLE: QUANTUM DECOHERENCE IN A PHONONIC GW DETECTOR

In this section we apply the general techniques derived above for quantifying the quantum decoherence of phonons of BECs to the specific example of a three-dimensional version of the phononic GW detector proposed in [27, 28]. We consider this to be a first step in the investigation of the quantum decoherence of this detector, which has so far been formulated with a one-dimensional BEC. This detector assumes an initial two-mode squeezed state of phonons that is subsequently modified by a GW in a measurable way. Methods for squeezing phonon states in experiments include introducing measurement back action under weak continuous probing [105], utilizing Beliaev damping [106], or implementing acoustic versions of Hawking radiation [38] or the dynamical Casimir effect [72, 73]. In [28], the effect of finite temperature on the performance of the device was analysed and was found to be negligible. However, the quantum decoherence of the phononic states at various phononic frequencies and temperatures was not considered. Here we attempt to obtain a rough estimate for how the device would be affected by quantum decoherence at high and low frequencies and temperatures by assuming a three-dimensional rather than one-dimensional BEC, and an initial single-mode squeezed state rather than a two-mode squeezed state.

The detector should, in principle, be able to measure GWs using phonons with frequencies around  $0.1 - 10$  kHz [27], and we therefore consider quantum decoherence over this range of phonon frequencies. From the evolution of the purity and nonclassical depth, we estimate the time scale of quantum decoherence of the phononic system. Additionally, we also analyse the relaxation of the system from the evolution of the average occupation, and characterize the evolution of the full state by further determining the evolution in squeezing.

### A. Phonons with $\mathcal{O}(\text{kHz})$ frequencies

The most interesting phonon frequencies for the GW detector are likely to be around  $\mathcal{O}(\text{kHz})$  since this allows for the detection of GWs with frequencies that are just beyond what LIGO can currently search for [27, 28, 107], and predicted signals include spinning neutron stars or neutron star mergers, which could inform us about the equation of state of such stars. The optimal strain sensitivity of the detector for this frequency range was obtained for an initial two-mode squeezed state with squeezing  $r = 10$ . So that these very high frequencies still correspond to phonons ( $\hbar\omega_q \ll mc_s^2$ ), a  $^{87}\text{Rb}$  BEC was assumed, where a temperature of 0.5 nK has been reached in a trapped system [37, 108], with a large density such that a speed of sound of  $c_s = 10^{-2} \text{ ms}^{-1}$  can be achieved. At these very high frequencies,  $k_B T \ll \hbar\omega_q$  is likely to be satisfied for temperatures theoretically up to the critical temperature of condensation of the Bose gas. From Section II B, this implies that Beliaev damping will dominate over Landau damping for this phonon mode in a uniform three-dimensional BEC, and the total damping rate  $\gamma$  for the phonons can be approximated as (51).

Figure 1 illustrates the time evolution of the purity, nonclassical depth, squeezing and mean occupation of a single-mode squeezed vacuum phonon state with  $r = 10$  and frequency  $\omega_q = 10 \times \omega_1$  in the BEC assumed in [27] and described above. Here  $\omega_1$  is the fundamental frequency of the trap, which is taken to be  $\omega_1/2\pi = 500 \text{ Hz}$ . For comparison, Figure 1 also describes the evolution of a thermal squeezed state (thermalized to the given BEC temperature), and a coherent state. All the states have been chosen such that the initial average occupations are equivalent.

From Figure 1 the squeezed states undergo a minimum of purity before asymptotically relaxing to the state of the environment, which is approximately in its vacuum state. The time  $t_{\min}$  at which the minimum is attained is given by (57) and, as discussed in Section III A, provides a good characterization of the decoherence time of such squeezed states [85]. This time is approximately given by  $t_{\min} \approx \ln 2/\gamma$  for highly squeezed states, which is approximately just the half-life of the state and is independent of temperature. This is to be compared with the time at which the non-classical depth reaches zero, which is approximately given by  $t_{\tau=0} \approx 2\beta_q/\gamma$ .

The quantum decoherence time, as characterized by  $t_{\min}$ , is around 6 s for this high frequency phonon mode. This quantum decoherence time would occur even at absolute zero, and decreases to around 0.6 s at 100 nK. A quantum decoherence time of 6 s would result in a loss of just one order of magnitude in strain sensitivity in comparison to that calculated using a phonon lifetime of 2000 s, which was assumed in [27]. This still allows for a sensitivity that improves upon that of advanced LIGO at the upper end of its frequency range (around  $7 \times 10^3 \text{ Hz}$ ) [27, 107].

### B. Phonons with $\mathcal{O}(0.1 \text{ kHz})$ frequencies

With high frequency phonon modes as considered in the previous sections, the GW detector would theoretically remain in the so-called quantum regime  $k_B T \ll \hbar\omega_q$  for temperatures approaching the critical temperature of condensation. However, if the detector were used instead to measure GWs with frequencies an order of magnitude smaller, then it would no longer operate in the quantum regime for temperatures higher than about 25 nK. In this case, Landau damping dominates over Beliaev damping and, for the BEC assumed in the previous section, the total damping rate  $\gamma$  for the phonons can be approximated as (53), which scales with temperature as  $T^4$ .

In order to illustrate the thermal regime in comparison to the quantum regime, the time evolution of the purity, nonclassical depth, squeezing and average occupation is illustrated in Figure 2 with the BEC setup assumed in the previous section but with phonon frequency  $\omega_q = 10 \times \omega_1$ , where now  $\omega_1/2\pi = 50 \text{ Hz}$ , and temperature  $T = 100 \text{ nK}$ . In comparison to the quantum regime, the time at which the squeezed states undergo a minimum of purity is now inversely proportional to  $T^4$ , and occurs at around 0.2 s for  $T = 100 \text{ nK}$ , whereas the nonclassical depth reaches zero at a time scale around four times shorter.

In contrast, for temperatures lower than 25 nK we recover the quantum regime where the phonon decoherence time would now be very high for the BEC assumed in [27] so that the performance of the device would instead be expected to be limited by the lifetime of the BEC. Furthermore, at  $T = 45 \text{ nK}$  we recover the same quantum decoherence time of 6 s as in the previous section for  $T = 0.5 \text{ nK}$ . Therefore, quantum decoherence of phonons is not expected to be a limiting factor for the performance of the device at low frequencies and temperatures.

### C. Phonons with $\mathcal{O}(10 \text{ kHz})$ frequencies

In [27], phonon frequencies of  $\omega_q/2\pi = 50 \text{ kHz}$  were also considered. For such frequencies, we can again approximate the total damping rate by (51) since the BEC will be operating in the quantum regime as in Section IV A. Figure 3 illustrates the time evolution of the purity, nonclassical depth, squeezing and mean occupation of a single-mode vacuum squeezed state with  $r = 10$  and frequency  $10 \times \omega_1$ , where now  $\omega_1/2\pi = 5000 \text{ Hz}$  as assumed in [27]. Again, for comparison, Figure 3 also describes the evolution of a thermal squeezed state (thermalized to the given BEC temperature), and a coherent state.

The quantum decoherence time, as characterized by  $t_{\min}$ , is around  $6 \times 10^{-5} \text{ s}$  for this very high frequency phonon mode. Such a small time scale suggests that quantum decoherence is likely to act as a limitation on the operation of the device for very high frequency phonon modes. In particular, from the derivation in [27], the strain sensitivity of the detector is roughly proportional to  $1/\sqrt{t}$ . This is due to a quadratic dependence on



## Frequency 5 kHz, Temperature 0.5 nK

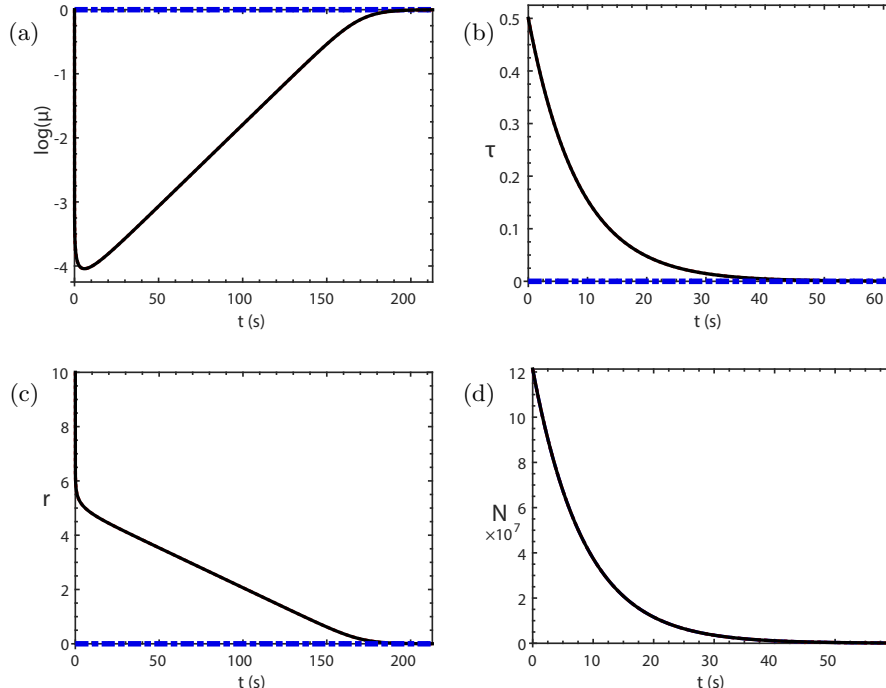


FIG. 1. Time evolution of (a) Purity  $\mu$ ; (b) Nonclassical depth  $\tau$ ; (c) Squeezing parameter  $r$ ; and (d) Average occupation  $N$ . A  $^{87}\text{Rb}$  BEC with  $c_s \sim 10^{-2} \text{ ms}^{-1}$  and temperature 0.5 nK was assumed, with the single phonon mode having frequency  $\omega_q/2\pi = 5 \text{ kHz}$ . The blue dot-dash line is an initial coherent state, and the black solid line represents a vacuum squeezed state with  $r = 10$ . A thermal squeezed state (thermalized to the given temperature) was also plotted but this does not visibly deviate from the black solid line of a vacuum squeezed state since the temperature is very low. All initial states were chosen such that the initial average occupation is the same, and so all lines overlap for subfigure (d).

$t$  coming from the Quantum Fisher Information for how long the phonons interact with the GWs, and the fact that a smaller  $t$  could allow for a faster repetition rate of the experiment. The latter of course assumes that the experiment can be performed in time  $t$ , which may not be the case for very small values. For example, to perform a measurement similar to that used to investigate the dynamical Casimir effect in a BEC [73], a time of  $t = 6 \times 10^{-5} \text{ s}$  would not provide enough time for the phonons to escape a condensate with length  $1 \mu\text{m}$ .

However, it should be emphasized that this estimate for the quantum decoherence time of the GW detector was calculated under various assumptions. In particular, in these quantum decoherence calculations a three-dimensional uniform BEC is assumed, whereas the GW detector was originally investigated in a one-dimensional setting, which would be expected to increase the quantum decoherence time since phonon interactions are likely to be suppressed in lower dimensional BECs [109–117]. It is therefore possible that quantum decoherence may not be such a limiting factor for the GW detector if it is built as a one-dimensional system as originally proposed in [27, 28]. A continuum of modes has also been assumed, but the GW has so far been looked at for a fairly discrete spectrum (a phonon frequency of 50 kHz was assumed but with a spectrum with levels that are  $\mathcal{O}(\text{kHz})$

apart), which would again likely result in the extension in this estimated decoherence time due to the reduction in phase space. Additionally, the effect of the GW itself was neglected in these calculations which, for such high frequency phonon modes, would have frequencies of  $\mathcal{O}(10^5 \text{ Hz})$ .<sup>5</sup>

Now that an estimate for the quantum decoherence time has been calculated for this frequency range we can also begin to investigate how the detector could be modified in order to increase this time. For example, one option might be to increase  $\omega_1$  such that the initial state involves a lower energy mode of the cavity for which Beliaev damping will be suppressed. However, this would require modifications to  $c_s$  and the trap geometry, such as increasing the speed of sound and reducing the effective size of the trap, which could also be challenging experimentally. Other options to increase the quantum decoherence time could be to squeeze the environment [85] and to utilise lower dimensional traps as described above.

<sup>5</sup> How phonon interactions affect the squeezing created by the dynamical Casimir effect, which is related to how the GWs create coherence of phonons [27], has been recently studied in [118].

## Frequency 500 Hz, Temperature 100 nK

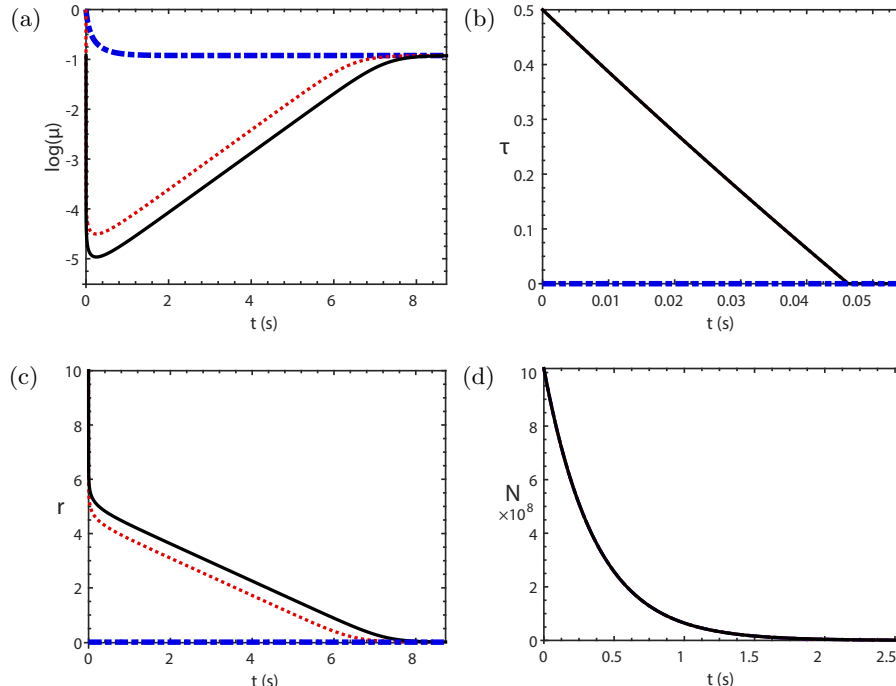


FIG. 2. Time evolution of (a) Purity  $\mu$ ; (b) Nonclassical depth  $\tau$ ; (c) Squeezing parameter  $r$ ; and (d) Average occupation  $N$ . A  $^{87}\text{Rb}$  BEC with  $c_s \sim 10^{-2} \text{ ms}^{-1}$  and temperature 100 nK was assumed, with the single phonon mode having frequency  $\omega_q/2\pi = 0.5 \text{ kHz}$ . The blue dot-dash line is an initial coherent state, the red dotted line is an initial thermal squeezed state (thermalized to the given temperature) with  $r = 10$ , and the black solid line is an initial vacuum squeezed state with  $r = 10$ . All initial states were chosen such that the initial average occupation is the same. Note that the nonclassical depth of the thermal and squeezed states is also approximately the same for such highly squeezed states, and so only the solid black line and blue dot-dash line are seen in subfigure (c).

## V. SUMMARY AND OUTLOOK

We have investigated the quantum decoherence time of phononic excitations of an isolated BEC in the Born-Markov approximation and assuming that the phonon states are Gaussian. The results can be used to assess the resourcefulness of phonons of BECs as carriers of quantum information. In particular, we have estimated the quantum decoherence time of phonons for a three-dimensional version of the GW detector that was proposed in [27], and found that this will only considerably affect the performance at very high phonon frequencies.

Although this work has been theoretical, it should also be possible to estimate the quantum decoherence of phonons experimentally. In particular, the general results of Section III should be applicable to generic BEC setups, not just the GW detector presented in Section IV, with the results just depending on a few experimental parameters such as temperature and frequency of the phonon mode. One possible way to measure the quantum decoherence time of a squeezed single-mode phonon state would be to determine the time at which the state's purity reaches a minimum. Purity is a non-linear function of the state's density operator and so is not related to the expectation value of a single-system Hermitian operator

or a single-system probability distribution that would be obtained from a positive operator-valued measure [100]. However, if the full quantum state of the system is known then the purity can be determined. For Gaussian states this would mean having to determine their first two statistical moments, which can be measured by the joint detection of two conjugate quadratures via heterodyne and multi-port homodyne detection schemes from quantum optics [100]. Similar methods have also been discussed for phonons of BECs in measurements of entanglement or squeezing [38, 106, 119–124], which could also likely be tailored to study purity.

We investigated the quantum decoherence of phonons of BECs by analysing the evolution of the purity and nonclassical depth of a single-mode system. It would, however, also be instructive to determine the evolution of proper coherence measures [125–129], which have recently been applied to infinite dimensional systems and Gaussian states [130, 131], for an analysis of quantum decoherence. For multi-mode states, another useful measure for loss of quantumness of a state is the evolution of entanglement [85]. Entanglement has recently been observed for phonon states in the emission of the acoustic analogue of Hawking radiation [38], potentially allowing for future studies into how the entanglement degrades

## Frequency 50 kHz, Temperature 0.5 nK

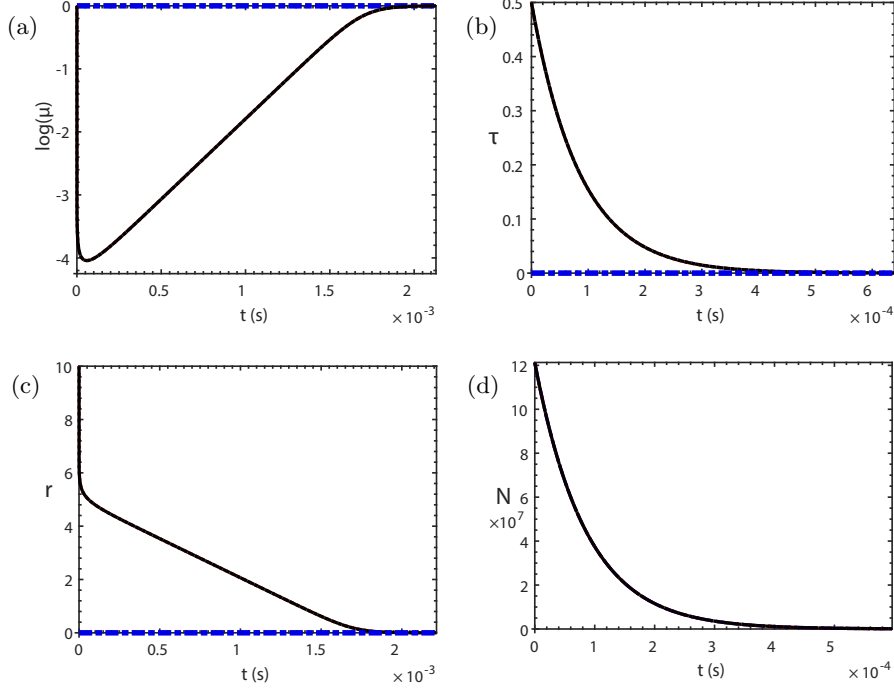


FIG. 3. Time evolution of (a) Purity  $\mu$ ; (b) Nonclassical depth  $\tau$ ; (c) Squeezing parameter  $r$ ; and (d) Average occupation  $N$ . A  $^{87}\text{Rb}$  BEC with  $c_s \sim 10^{-2} \text{ ms}^{-1}$  and temperature 0.5 nK was assumed, with the single phonon mode having frequency  $\omega_q/2\pi = 50 \text{ kHz}$ . The blue dot-dash line is an initial coherent state, and the black solid line represents a vacuum squeezed state with  $r = 10$ . A thermal squeezed state (thermalized to the given temperature) was also plotted but, as described in Figure 1, this is not visible. All initial states were chosen such that the initial average occupation is the same.

with time. Understanding this de-entanglement processes could dictate what is possible to measure in analogue experiments, and perhaps provide potential clues to the information paradox in black hole physics [132, 133]. We leave such proposals, however, to future work.

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